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Classification of Energy Flow Observables in Narrow Jets

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We present a classification of energy flow variables for highly collimated jets. Observables are constructed by taking moments of the energy flow and forming scalars of a suitable Lorentz subgroup. The jet shapes are naturally arranged in an expansion in both angular and energy resolution, allowing us to derive the natural observables for describing an N-particle jet. We classify the leading variables that characterize jets with up to 4 particles. We rediscover the familiar jet mass, angularities, and planar flow, which dominate the lowest order substructure variables. We also discover several new observables and we briefly discuss their physical interpretation.

I. INTRODUCTION

Analyzing the substructure of highly-boosted massive jets ($p_J \gtrsim 500$ GeV) has proved to be useful for distinguishing new physics signals from the QCD background. For instance, jets originating from boosted electroweak gauge bosons [1], tops [2, 3], Higgs [4], and even new physics particles [5, 6] are all interesting targets of searches conducted at the Tevatron and the LHC experiments, and it is therefore important to be able to distinguish them from QCD jets. For recent reviews on substructure techniques, experimental status and new physics searches see [7] and references therein.

One way to characterize jet substructure is to consider observables which are functions of the energy flow within the jet, namely the energy distribution $\varepsilon(\vec{x})$ as measured by the detector, where $\vec{x}$ are the coordinates on the detector surface. Examples of such observables include the jet mass and the angularities [8][1],

$$\tau_a \equiv \frac{1}{E_J} \sum_{i \in \text{particles}} E_i \left(\sin\theta_i\right)^a \left(1 - |\cos\theta_i|\right)^{1-a} , \tag{1}$$

where E_J is the jet energy, E_i is the energy of particle i, θ_i is the angle relative to the jet axis, and a is a real parameter restricted to $a \leq 2$ to ensure IR safety. Another example is planar flow [9] (see also [10]), which we discuss below in detail.

While many observables have been defined, so far there has been no systematic way to classify them or to construct all observables at a given order in the detector resolution. The goal of the present work is to suggest such a classification for the case of narrow jets, as we now describe.

An energy flow observable needs to meet several criteria. Ideally, it should be Lorentz invariant and IR-collinear (IRC) safe. In practice it is difficult to find observables which are Lorentz invariant [10], the jet mass being an exception.

[1] Here we use a different normalization than that of [8].

We may however relax this requirement as follows. For highly-boosted, narrow jets with fixed 4-momentum the cross section factorizes at leading order into the hard process and the jet function (in which we are interested). Since the jet momentum is an input parameter of the jet function, it is enough to consider observables which are invariant under the subgroup of Lorentz that doesn't change the jet momentum. For a massive jet, this little group is isomorphic to SO(3), as can be seen by boosting to the jet rest frame.

Another property of a jet is that it is contained in a cone of radius $R \ll 1$ around its axis, and we should further restrict ourselves to Lorentz transformations that leave the cone invariant[2]. The subgroup that stabilizes both the jet momentum and the cone is SO(2), the rotations around the jet axis.

In the present paper we focus on highly-collimated jets, working at leading order in the small cone size $\theta_i < R \ll 1$. We will further assume that all the particles that make up the jet are massless. Under these assumptions, we propose a complete classification of energy flow observables which are IR safe and Lorentz invariant, up to Lorentz transformations that change the jet momentum and the cone. Furthermore the observables may be naturally arranged as an expansion in terms of the energy resolution and the cone size (or alternatively the energy and angular resolutions).

In the narrow cone approximation the experimental calorimeter information of the jet is fully specified by the energy distribution on a surface perpendicular to the jet axis, $\varepsilon(\vec{x})$, with $\vec{x} \equiv (x_1, x_2)$ corresponding to a set of coordinates on the two dimensional jet surface. To obtain the classification, we begin by describing a given energy distribution $\varepsilon(\vec{x})$ in terms of its moments,

$$I_{i_1 \ldots i_n} = \int d^2x \, \varepsilon(x) \, x_{i_1} \cdots x_{i_n} \, .$$

Observables are then constructed by taking products of these moments, and forming SO(2) scalars by contracting their indices. Invariance under boosts along the jet axis can be achieved by adding a factor of E_J raised to some power. Since our moments form a straightforward generalization of angularities with $a \leq 2$, they are manifestly IR safe.

Moments of higher rank are of higher order in the cone size R, since $x \propto \theta$, the opening angle. Moreover, since $I \sim \epsilon(x)$, observables that are products of a larger number of moments are more sensitive to errors in energy measurement (we will be more precise about this point below). This allows us to arrange the observables in an expansion by energy resolution and cone size as will be shown in section III. At the lowest orders in this expansion we find the jet mass and the angularity with $a = -2$. At the next order we find the angularity with $a = -4$ and the planar flow. Yet higher orders produce the whole tower of angularities with $a = -2k$, as well as many new observables.

The paper is organized as follows. In the next section we review the definition of planar flow, showing how it naturally leads to the expansion of the energy distribution in terms of moments. In section III we define the moment expansion, find the first few observables, and precisely identify the small parameters in which we expand. In section IV we turn to a classification of the leading observables that characterize jets with up to 4 particles. In section V we present an alternative approach to the study of jet substructure, based on an expansion in terms of orthogonal functions. Finally, in section VI we analyze one of the new observables discovered in the classification, developing techniques that may be applied to other observables as well. Section VII contains a few comments on the application of this formalism to the case of hadron colliders and our conclusions.

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[2] Alternatively, we may describe the jet as its 4-momentum plus a detector surface of small area, perpendicular to the jet axis. The little group should leave the momentum and surface invariant. Moreover, even if we will refer to the jet cone throughout the paper, our conclusions are not restricted to jet cone algorithms. For us it is sufficient that there exist a number $\bar{R} < 1$ such that each jet is contained in cones of radius $\bar{R}$, which is also generically the case for sufficiently hard jets found with sequential recombination algorithms. Indeed, for narrow massive jets, it was shown that the angularity distributions are similar for a wide range of jet algorithms [11].

II. MASS, PLANAR FLOW, AND THE SECOND MOMENT

Planar flow is defined in terms of the matrix I_w, with components[3]

$$I_w^{kl} = \sum_{i\in\text{particles}} E_i \frac{p_{i,k}^{\perp}}{E_i}\frac{p_{i,l}^{\perp}}{E_i} \approx \sum_{i\in\text{particles}} E_i\,\theta_i f_k(\phi_i)\,\theta_i f_l(\phi_i)\,,$$

where $p_\perp$ is the particle momentum in the detector plane, ϕ is the azimuthal angle, and $f_1(\phi) = \cos(\phi)\,, f_2(\phi) = \sin(\phi)\,$. The approximation is made under the narrow cone assumption. Planar flow [11] is then defined as

$$\text{Pf} = \frac{4\det I_w}{(\text{Tr}\, I_w)^2}\,.$$

One can easily verify that $0 \le \text{Pf} \le 1$, that it vanishes when the energy distribution lies on a line, and that it is maximal for isotropic distribution.

I_w also enters in the definition of the jet mass. From $m_J^2 = p_J^2$, $p_J = \sum_i p_i$ it is easy to see that, in the narrow cone approximation,

$$\frac{m_J^2}{E_J} \approx \sum_{i\in\text{particles}} E_i\theta_i^2 = \text{Tr}\, I_w\,. \tag{2}$$

The narrow cone approximation ensures also that $m_J \ll E_J$.

Now, let $x_k(\theta,\phi) = \theta f_k(\phi)$ denote a coordinate system on the detector plane. Then we may write I_w as a sum over detector cells, where we weigh each cell by its total collected energy $E(x)$, namely

$$I_w^{kl} = \sum_{n\in\text{cells}} E\left(x^{(n)}\right) x_k^{(n)} x_l^{(n)}\,,$$
$$E(x) = \sum_{i\in\text{particles}} E_i\,\delta_{\theta_i,\theta(x)}\delta_{\phi_i,\phi(x)}\,.$$

One can define a normalized version of this tensor, $I_{kl} = I_w^{kl}/E_J$, where E_J is the jet energy. For the sake of clarity let us rewrite the above expression in integral form,

$$I_{kl} = \int d^2x\,\varepsilon(x)\,x_k x_l\,, \tag{3}$$

where $\varepsilon(x)$ is the continuous, normalized energy distribution, given by

$$\varepsilon(x) = \frac{1}{E_J}\sum_{n\in\text{cells}} E\left(x^{(n)}\right)\delta(x - x^{(n)})\,.$$

Note that

$$\int d^2x\,\varepsilon(x) = 1\,. \tag{4}$$

[3] We use a different normalization for I_w than that of [11], without affecting the planar flow definition.

Equations (3) and (4) are quite suggestive. The function $\varepsilon(x)$ encodes all the information about the jet structure and it is the object we want to characterize. Now, we have just seen that I_{kl} is its second moment and it gives rise to two interesting physical observables— the jet mass and the planar flow. It is well known that all the information of $\varepsilon(x)$ is encoded in its moments. It is therefore plausible that expressing the energy distribution $\varepsilon(x)$ in terms of its moments would provide a natural way to derive observables with the desired properties. One can also expand the function $\varepsilon(x)$ in a set of orthogonal functions and characterize it in terms of its expansion coefficients. SO(2) invariance will be ensured by taking suitable coefficient combinations that are singlets under rotation. The two approaches are complementary and can provide different insight on the properties of $\varepsilon(x)$. These are the ideas that we will explore in the remainder of this work.

III. EXPANSION IN MOMENTS

The n-th moment I_n of the energy distribution $\varepsilon(x)$ is defined by

$$I_{k_1,\ldots,k_n} \equiv \int d^2x\, \varepsilon(x)\, x_{k_1} \cdots x_{k_n} = \frac{1}{E_J} \sum_{i\in\text{particles}} E_i\, x^{(i)}_{k_1} \cdots x^{(i)}_{k_n} \,.$$

The zeroth moment is (4). The first moment is the expectation value, or dipole. It is set to zero by the requirement that the total transverse momentum of the jet vanishes, a state which can be reached by rotating the jet. This fixes the origin of the detection plane, which in turn determines the jet axis. We have then

$$I_0 = 1\,, \quad I_1 = 0\,.$$

The first non-trivial moment is therefore I_2, as expected. We are looking to define observables that are invariant under the little group SO(2), the Lorentz subgroup that doesn't change the jet momentum or the cone. SO(2) has two independent invariant tensors, δ_{ij} and ϵ_{ij}. The only SO(2) scalar that is linear in I_2 is the normalized version of (2), namely

$$I_{ii} \approx \frac{m_J^2}{E_J^2}\,.$$

Next, consider a tensor product $I_2 \otimes I_2$. There are three nontrivial scalars one may construct,

$$I_{ii}I_{jj}\,, \quad I_{ij}I_{ij}\,, \quad \epsilon_{ij}\epsilon_{kl}I_{ik}I_{jl}\,.$$

Of these, only two are independent, since

$$\epsilon_{ij}\epsilon_{kl}I_{ik}I_{jl} = 2(I_{ii}I_{jj} - I_{ij}^2) = 2\det I\,.$$

Also, the first scalar, $I_{ii}I_{jj}$, factorizes in lower-rank scalars. We therefore find only one new observable, $\det(I)$, which is an un-normalized version of planar flow.

Before proceeding with the expansion, let us clarify in what sense the moment expansion is an expansion in small parameters. Planar flow, composed of $I_2 \otimes I_2$, is apparently of a higher order than the mass squared, composed of a single I_2. To quantify this statement, first note that energy distribution is constrained to lie within a small cone of radius $R \ll 1$. Since $I_n \sim x^n \sim \theta^n$, we have that I_n scales as R^n. As an expansion in R, planar flow is of order 4 while the jet mass squared is of order 2.

Another small parameter we may consider is the angular resolution $\Delta\theta$, which is ultimately limited by the calorimeter cell size. Including its effect gives $I_n \sim (\theta \pm \Delta\theta)^n = \theta^n \pm n\Delta\theta\,\theta^{n-1} + \cdots$. The error on

R \ $\Delta\varepsilon$	1	2	3	4
2	I_2	-	-	-
4	I_4	$(I_2)^2$	-	-
6	I_6	$I_2 I_4, (I_3)^2$	$(I_2)^3$	-
8	I_8	$I_2 I_6, I_3 I_5, (I_4)^2$	$I_2(I_3)^2, (I_2)^2 I_4$	$(I_2)^4$

TABLE I: Moment products that correspond to given orders in energy resolution $\Delta\varepsilon$ and cone size R. Observables are constructed by contracting these products in various ways.

the value of an observable due to the finite angular resolution is therefore proportional to the total rank of moments composing this observable. Planar flow has error $\sim 4\Delta\theta$, while the mass squared has error $\sim 2\Delta\theta$. Regardless of which small angular parameter we choose, measuring a moment of higher rank requires a more accurate detector.

As for the energy, the only small parameter is the energy resolution $\Delta\varepsilon$. A single moment is proportional to $\varepsilon(x)$, so for a given observable the error due to energy resolution will increase with the number of moments that make up the observable. Planar flow will tend to have a larger error than the mass.

IV. THE FIRST FEW ORDERS

We are now in a position to classify all the jet energy flow observables.

Table I lists the outer products that appear in the first few orders of the expansion. Note that there are no observables with an odd power of R since we cannot fully contract an odd number of SO(2) indices. Each outer product may be contracted in several different ways, giving rise to different observables. However, algebraic relations will reduce the total number of independent contractions.

On top of this expansion it is useful to consider jets that consist of a given number of particles (or detector cell towers). A jet of N particles is characterized by $3N-4$ variables, corresponding to the number of particle momentum components, minus the jet axis, the jet energy, and the SO(2) angle associated to the overall rotation around the jet axis.[4] For given N we can then identify the lowest order, Lorentz-invariant observables that characterize such jets. Let us list the observables for jets with up to 4 particles.

The two lowest order observables are

$$I_{ii} = \frac{1}{E_J} \sum_{i\in\text{particles}} E_i \theta_i^2 \approx \frac{m_J^2}{E_J^2} \,,$$

$$I_{iijj} = \frac{1}{E_J} \sum_{i\in\text{particles}} E_i \theta_i^4 \approx 8\,\tau_{-2} \,.$$

Therefore, the natural observables for describing a two-particle jet are the mass and the angularity τ_{-2}.

To describe 3-particle jets we need 3 more observables in addition to the mass and τ_{-2}. As shown above, the outer product $(I_2)^2$ has one independent contraction, $\epsilon_{ik}\epsilon_{jl} I_{ij} I_{kl} \sim \det I$, the planar flow. Next we

[4] This is at leading order. Soft phenomena, such as color connection with other jets in the event, will generally break this symmetry.

have I_6 which has one contraction,

$$I_{iijjkk} = \frac{1}{E_J} \sum_{i \in \text{particles}} E_i \theta_i^6 = 32\, \tau_{-4} \,,$$

corresponding to another member of the angularity family with higher a. Generally, I_{2n} with $n > 1$ has only one contraction, $I_{i_1 i_1 \cdots i_n i_n} \sim \sum_i E_i \theta_i^{2n}$, which corresponds to an angularity with $a = 2(1-n)$.

At the next order we find $I_2 I_4$, $(I_3)^2$, and I_8, with the following independent contractions:

$$\begin{aligned} I_2 I_4 &: \ \epsilon_{ij}\epsilon_{kl} I_{ik} I_{jlmm} \,, \ \epsilon_{ij} I_{ik} I_{jkll} \\ (I_3)^2 &: \ \epsilon_{ij}\epsilon_{kl} I_{ikm} I_{jlm} \,, \ I_{ijk} I_{ijk} \\ I_8 &: \ I_{iijjkkll} \end{aligned}$$

The additional contractions $I_{ij} I_{ijkk}$, $I_{ijj} I_{ikk}$ can be shown to be linearly dependent on these. Any combination of them can be chosen as the remaining observable for 3-particle jets. In lumping I_8 with the rest we assumed for simplicity that energy and angular measurements have comparable weight in the small parameter expansion. Of course, if this assumption is not true, I_8 may be preferred or disfavored in comparison with the other contractions.

Note that an observable—such as planar flow—which includes one or more ϵ symbols vanishes when all particles are on a line.[5] Since two particles always lie on a line, these observables contribute at leading order to 3-particle jets. They can therefore be used to distinguish QCD jets, which favor 2-parton configurations, from e.g. top jets that favor 3-body decays.

Finally, 4-particle jets are described by eight variables. So far we found nine leading order observables,

$$\begin{gathered} I_{ii} \,, \quad I_{iijj} \,, \quad \epsilon_{ij}\epsilon_{kl} I_{ik} I_{jl} \,, \quad I_{iijjkk} \,, \quad \epsilon_{ij}\epsilon_{kl} I_{ik} I_{jlmm} \,, \\ \epsilon_{ij} I_{ik} I_{jkll} \,, \quad \epsilon_{ij}\epsilon_{kl} I_{ikm} I_{jlm} \,, \quad I_{ijk} I_{ijk} \,, \quad I_{iijjkkll} \,. \end{gathered}$$

Of these, any leading eight can be chosen to describe the jet.

V. EXPANSION IN ZERNIKE POLYNOMIALS

One can expand $\varepsilon(x)$ in a series of orthogonal functions. Since $\varepsilon(r,\phi)$ is defined on a disc of radius R, perhaps the most convenient expansion is in terms of the Zernike polynomials [12], which form an orthogonal basis on the unit disc. They are defined by

$$Z_n^m(\rho,\phi) = R_n^m(\rho)\cos(m\phi) \,, \quad Z_n^{-m}(\rho,\phi) = R_n^m(\rho)\sin(m\phi) \,, \quad 0 \le \rho \le 1 \,,$$

where $0 \le m \le n$, $n-m$ even, and $R_n^m(\rho)$ are a set of polynomials of degree n respecting the orthogonality condition

$$\int_0^1 \mathrm{d}\rho \rho R_n^m(\rho) R_{n'}^m(\rho) = \frac{1}{2n+2} \delta_{n,n'} \,.$$

The orthogonality among different m's follows trivially from the orthogonality of the Fourier modes. This set of functions is widely used in optics, in particular in the study of optical aberrations where the expansion coefficients have simple geometrical meaning.

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[5] To see this, first rotate the line configuration to lie on the x_1 axis, and then note that the ϵ tensor forces an x_2 factor to appear. This factor vanishes wherever $\varepsilon(x) \neq 0$.

The expansion of the energy distribution is

$$\varepsilon(r,\phi) = \frac{a_{0,0}}{R^2} + \frac{1}{R^2}\sum_{n=1}^{\infty}\sum_{\substack{0\le m\le n,\\ n-m\,\text{even}}}\left[a_{n,m}R_n^m\left(\frac{r}{R}\right)\cos(m\phi) + a_{n,-m}R_n^m\left(\frac{r}{R}\right)\sin(m\phi)\right], \tag{5}$$

The conditions $I_0 = 1$ and $I_1 = 0$ fix $a_{0,0} = 1/\pi$, and $a_{1,\pm1}$ to vanish. One can further expand the moments defined in section III in terms of the $a_{n,m}$, finding that a moment of order r will be expressed as a linear combination of $a_{n,m}$'s with $n \le r$. Moreover, upon tracing over k of the r indices, m will be constrained to be $\le r-k$. An even/odd number of indices corresponds to m (and thus n) being even/odd.

The lowest order invariants like the mass, the angularities with $a = 2, 4$ and planar flow have the following expression in terms of the Zernike coefficients $a_{n,m}$:

$$\begin{aligned}
\frac{m_J^2}{E_J^2} &= \frac{\pi}{6}R^2\left(a_{2,0}+3a_{0,0}\right),\\
\tau_{-2} &= \frac{\pi}{240}R^4\left(a_{4,0}+5a_{2,0}+10a_{0,0}\right),\\
\tau_{-4} &= \frac{\pi}{4480}R^6\left(a_{6,0}+7a_{4,0}+21a_{2,0}+35a_{0,0}\right),\\
(1-\mathrm{Pf})\,\frac{m_J^4}{E_J^4} &= \frac{\pi^2}{36}R^4\left(a_{2,2}^2+a_{2,-2}^2\right).
\end{aligned}$$

In optics, the lowest order coefficients have been given names. In particular $a_{1,\pm1}$ is the tilt, $a_{2,0}$ is the defocus, $a_{2,\pm2}$ are the 0° and 45°-astigmatism (respectively), $a_{3,\pm1}$ is the coma, and $a_{4,0}$ is the spherical aberration. The optical analogy may provide us with additional geometrical understanding of what is being probed by the jet shapes defined in the previous sections.

VI. ANALYSIS OF $\epsilon I_2 I_4$

We now try to gain some intuition regarding one of the new observables found in section IV, $\mathcal{O} \equiv 2\epsilon_{ij}I_{ik}I_{jkmm}$. Note that any observable which includes an odd number of ϵ_{ij} symbols is a pseudo-scalar with respect to parity on the detector plane. It therefore vanishes for any energy distribution that is symmetric under reflection through an arbitrary axis.

Let I_4' be the I_4 moment traced once, namely I_{kkij}. Both I_2 and I_4' are real, symmetric matrices, so we can write them in terms of the Pauli matrices,

$$\begin{aligned}
I_2 &= I_{2,0}\frac{\sigma^0}{2} + I_{2,1}\frac{\sigma^3}{2} + I_{2,2}\frac{\sigma^1}{2},\\
I_4' &= I_{4,0}\frac{\sigma^0}{2} + I_{4,1}\frac{\sigma^3}{2} + I_{4,2}\frac{\sigma^1}{2}.
\end{aligned}$$

The observable can then be written as

$$\mathcal{O} = 2\epsilon_{ij}I_{ik}I_{jkmm} = 2\,\mathrm{Tr}(I_2\epsilon I_4') = \epsilon_{ij}I_{2,i}I_{4,j}.$$

Note that $\mathcal{O}$ doesn't depend on the σ^0 components (the trace). Treating $I_{2,i}, I_{4,j}$ as $d = 2$ vectors, we see that

$$\mathcal{O} = \vec{I}_2 \times \vec{I}_4, \tag{6}$$

where the product is the cross product in $d=2$ which produces a pseudo-scalar.

To get further insight, let us compute the components of these vectors. We do this for a general moment I'_{2k} with all indices traced except two. It is expanded in Pauli matrices just like I_2, I'_4 above, and we denote its corresponding $d=2$ vector by I^i_{2k}. We write the result using polar coordinates, $x_1 = r\cos(\phi)$, $x_2 = r\sin(\phi)$.

$$
\begin{aligned}
I^1_{2k} &= \mathrm{Tr}(I'_{2k}\sigma^3) = I'_{2k,11} - I'_{2k,22} = \int_0^R dr \int_0^{2\pi} d\phi\, \varepsilon(r,\phi)\, r^{2k+1} \cos(2\phi)\,, \\
I^2_{2k} &= \mathrm{Tr}(I'_{2k}\sigma^1) = 2I'_{2k,12} = \int_0^R dr \int_0^{2\pi} d\phi\, \varepsilon(r,\phi)\, r^{2k+1} \sin(2\phi)\,.
\end{aligned}
$$

Plug the Zernike expansion (5) of the energy distribution in $\vec{I}_{2k}$. The ϕ integral picks out the $m=\pm 2$ modes, and the radial integral picks up a linear combination of $a_{2k,\pm 2}, \ldots, a_{2,\pm 2}$, with $+$ sign for the cosine and $-$ sign for the sine. Specializing for the cases $k=1,2$ we find:

$$
\vec{I}_2 = \frac{\pi}{6} R^2 \begin{pmatrix} a_{2,2} \\ a_{2,-2} \end{pmatrix}\,, \quad \vec{I}_4 = \frac{\pi}{40} R^4 \begin{pmatrix} a_{4,2} + 5a_{2,2} \\ a_{4,-2} + 5a_{2,-2} \end{pmatrix}\,.
$$

The observable (6) can now be written as

$$
\mathcal{O} = \frac{\pi^2}{240} R^6 \det \begin{pmatrix} a_{2,2} & a_{4,2} \\ a_{2,-2} & a_{4,-2} \end{pmatrix}\,, \tag{7}
$$

where we have dropped the piece in $\vec{I}_4$ which is proportional to $\vec{I}_2$ because of the usual properties of the determinant.

This shows that, in this expansion of $\varepsilon(x)$, $\mathcal{O}$ depends only on the $m=\pm 2$ Fourier modes for the angular part and on two specific combinations of the even, $n=2,4$ Zernike modes.

Finally, let us define the normalized observable

$$
\mathcal{O}_{\mathrm{n}} = \frac{\vec{I}_2 \times \vec{I}_4}{|\vec{I}_2||\vec{I}_4|}\,,
$$

which assumes values in the range $[-1,1]$. This observable is maximal when $\vec{I}_2$ and $\vec{I}_4$ are orthogonal, and vanishes when $\vec{I}_2$, $\vec{I}_4$ are linearly dependent. This happens, for instance, when the energy distribution is invariant under reflection through some axis. Indeed, in that case we may rotate this axis to coincide with the x_1 direction, following which $a_{n,-2}=0$ since all the antisymmetric Fourier modes vanish. This agrees with the fact that this observable is a pseudo-scalar.

Moreover, I_2 and I'_4 differ only by a different weighing of the inner/outer part of the jets (due to the additional r^2 in I'_4). A symmetric matrix defines a characteristic ellipsoid (in the case of I_2 it is the ellipsoid of inertia in the detector plane). A non-zero $\vec{I}_2 \times \vec{I}_4$ determines by how much the principal axes of the ellipsoid rotate when we give extra weight to the outer portion of the jet. (In the language of optics it would correspond to measuring the relative orientation of the lowest and higher order astigmatism.) With this intuition it is also trivial to see that $\mathcal{O}_{\mathrm{n}}$ vanishes when all the particles lie at the same distance from the jet axis.

We will now briefly study some features of the $\mathcal{O}_{\mathrm{n}}$ distribution. While on an event-by-event basis $\mathcal{O}_{\mathrm{n}}$ will take both positive and negative values, in most cases the distribution will be symmetric around zero, yielding $\langle \mathcal{O}_{\mathrm{n}} \rangle = 0$ due to its pseudoscalar nature. A necessary condition for $\langle \mathcal{O}_{\mathrm{n}} \rangle \neq 0$ is that the

substructure of the jet is controlled by a parity violating interaction. However, the interesting study of whether and when $\langle\mathcal{O}_{\rm n}\rangle$ is non-zero is beyond the scope of this work; here we will merely present distributions for $|\mathcal{O}_{\rm n}|$ in a few characteristic kinematic scenarios. Since this observable vanishes for two-particle configurations, in Fig. 1 we present the distributions for three-particle final states. We include both the case of a pure one-to-three decay and the case of a three-body decay of particle A via two subsequent two-body decays, with an on-shell intermediate particle B (as would be in the case of top decays). In the latter case one can see the sharp edge, whose position is just determined by the ratio [6] m_B/m_A.

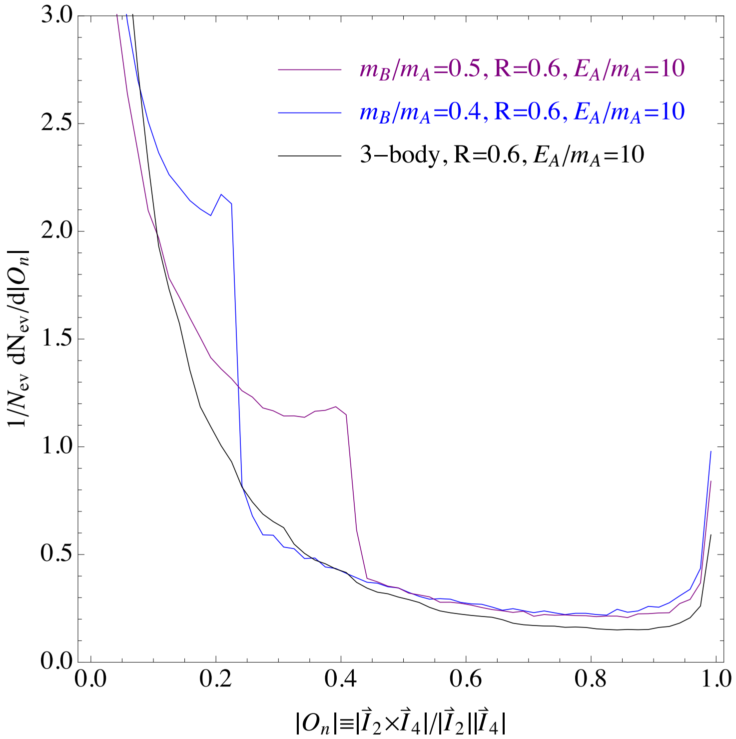

FIG. 1: The $\mathcal{O}_{\rm n}$ normalized distribution for various values of m_B/m_A, the mother particle boost was fixed to $E_A/m_A = 10$.

VII. DISCUSSION

While the study presented in this paper is directly applicable to e^+e^- machines, let us briefly discuss its applicability to hadronic colliders, such as Tevatron and the LHC. Since the partonic center of mass system is unknown in a hadronic collision, one clearly has to have a definition for the moments which is invariant under boosts along the beam axis. At leading order in the narrow cone approximation this is trivially achieved by using $\Delta\eta$ and $\Delta\phi$ (differences in pseudorapidity and in azimuthal angle) for the

[6] We checked that it is insensitive to the values of R and of the jet energy E_J, if varied within reasonable ranges.

x_1 and x_2 coordinates. In fact $\varepsilon(\vec{x})$ depends only on $\vec{x}$ and the ratios E_i/E_J. Since $E = p_T \cosh\eta$ one finds that $\varepsilon(\vec{x})$ is also invariant at leading order in the narrow cone approximation. Corrections start at $O(\tanh\eta_J \Delta\eta)$ and contaminate a moment of order r with moments of order $k > r$ (due to the additional powers of $\Delta\eta$). However, being of higher order, they will be suppressed by powers of the cone size R, rendering the contamination small.

Another difference with the e^+e^- case is in the structure of the intra-jet radiation. Indeed in terms of symmetries the presence of beam remnants which can be color-connected with the jet under study provide a specific direction, common to all events, of breaking of the SO(2) symmetry. While these effects have been widely studied in the literature in various contexts (see *e.g* [7, 13, 14]), it would still be interesting to see how they are represented in the language developed in this paper. While we will not embark here on a systematic study, we will sketch the analysis and outline a few results.

One can relax the requirement of the SO(2) invariance and still use the same formalism described in this paper to study these effects. In general there will be a set of directions corresponding to the beam axis, the directions of other jets in the event, etc. They identify two-dimensional unit vectors $\vec{n}_k$ in the detector plane that can be used to contract the indices of the energy flow moment tensors. To lowest order and considering for simplicity the case of a single $\vec{n}$ one has

$$\begin{aligned} &n^i n^j I_{ij}\,, \quad n_i \epsilon^{ij} I_{jk} n^k\,, \\ &I_{iij} n^j\,, \quad I_{iij}\epsilon^{jk} n_k\,. \end{aligned}$$

Note also that in our counting of how many independent observables are present for an N-particle system, SO(2) invariance eliminated one observable. Therefore, while the underlying event and soft radiation from color reconnection can contaminate all the jet shape observables presented in this paper, at leading order there is room for only one observable able to probe these effects via a systematic SO(2) non-invariance. This observable can be chosen to be any combination of the lowest order contractions such as the ones shown above. In particular it is worth noticing that $I_{iij}n^j$ is closely related to $\vec{t}\cdot\vec{n}$ where $\vec{t}$ is the pull variable defined in [15], the only differences being a different power of the cone size.

To conclude, we have presented an order-by-order classification of jet shape variables for narrow jets with massless constituents. At the first few orders we encountered familiar variables—jet mass, angularity, and planar flow—and at higher orders we found new observables. Specifically, we proposed several new observables that may be used to characterize 3- and 4-particle jets. While the classification is complete, the formalism we introduced does not provide much insight into the geometric nature of the observables (beyond a few simple properties), and for that we need additional tools. Expanding the energy distribution in terms of Zernike polynomials seems especially suited for this purpose, at least for the set of observables analyzed here. It will be interesting to analyze other observables in this way, and in particular the techniques used for studying $\epsilon I_2 I_4$ can be easily applied to other observables that involve the moments I'_{2k}. A more thorough study of the properties of these new observables, and the identification of processes for which they provide discriminating power, is an important topic left to future study.

The fact that the Zernike polynomials, which are commonly used in the field of optics, arise naturally in our jet shape analysis is obviously related to the fact that in both cases one is describing some distribution on a disc. One may hope however that the analogy between optics and jet substructure is deeper. If that is the case, further insight into the description of jet energy flow may be gained from the well-studied theory of aberrations.

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